M317 sec 2 Assignment 3 due Wed

1. Prove that if $\beta = \inf(A)$ exists, but $\beta \notin A$, then β must be an accumulation point for A. Use this fact to prove that every finite set contains its $\inf$.

Let $\beta = \inf(A) \notin A$. Then β is a lower bound for A and for every $\varepsilon > 0$, $\beta + \varepsilon$ is not a LB for A. Then there is some element $p \in A$ such that $\beta \le p < \beta + \varepsilon$. Since $\beta \notin A$ and $p \in A$, it can't be that $\beta = p$, so $\beta < p < \beta + \varepsilon$. But this implies that $\forall \varepsilon > 0$, $\mathring{N}_\varepsilon(\beta) \cap A \neq \emptyset$ so β is an accumulation point of A. Any finite set is composed entirely of isolated points so it has no accumulation points. Therefore the $\inf$ must belong to the set. In fact it is just the smallest element in the set.

2. Prove that if $\inf(A)$ exists then it is unique.

Let $\beta_1 = \inf(A)$ and $\beta_2 = \inf(A)$. Then β_1 is a lower bound for A and $\beta_1 \le \alpha$ for any other lower bound, α. In particular, $\beta_1 \le \beta_2$. By the same arguement, $\beta_2 \le \beta_1$. But then $\beta_1 = \beta_2$.

3. Prove that if A and B are closed, then $A \cap B$ is closed.

$A \cap B$ is closed if $(A \cap B)^C$ is open. We know that $(A \cap B)^C = A^C \cup B^C$ and since A and B are closed, it follows that A^C and B^C are open so that every point of A^C and every point of B^C is an interior point. Let p denote any point of $A^C \cup B^C$. Then p belongs to A^C or B^C, so suppose $p \in A^C$. Then p is an interior point of A^C so it is also an interior point of $A^C \cup B^C$. This shows that every point of $A^C \cup B^C$ is an interior point so $A^C \cup B^C$ is open and $A \cap B$ is closed.

4. Let $S_m = \sum_{n=0}^{m} r^n$, where $0 < r < 1$. Show that the set of values $A = \{S_1, S_2, \ldots\}$ is bounded above by the value $\sigma = \frac{1}{1-r}$.

We showed in class that $S_m - rS_m = 1 - r^{m+1}$, and this implies

$$S_m = \frac{1 - r^{m+1}}{1-r} < \frac{1}{1-r}.$$

5. To show that $\sigma = \sup(A)$, show that for every $\varepsilon > 0$, $\sigma - \varepsilon < S_m < \sigma$, if $m > \frac{\log[\varepsilon(1-r)]}{\log r}$.

Note that

$$0 < \sigma - S_m = \frac{r^{m+1}}{1-r}$$

Then

$$\frac{r^{m+1}}{1-r} < \varepsilon$$

$$\text{if} \quad r^{m+1} < \varepsilon(1-r),$$

For $0 < r, \varepsilon < 1$ we have that $0 < \sigma - S_m < \varepsilon$, if

$$m > \frac{\log[\varepsilon(1-r)]}{\log r}.$$